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A Note on the Distribution of Differences between Consecutive Prime Numbers

Lester D. Taylor
University of Arizona

Department of Agricultural and Resource Economics
College of Agriculture and Life Sciences
The University of Arizona

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Abstract

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Between Consecutive Prime Numbers

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The results reported in this note refer to the distribution of $z_n = p_n - p_{n-1}$ for the first three million prime numbers (p). The analyses of the note are almost purely statistical. The difference between consecutive prime numbers is treated as a random variable, and empirical frequency distributions are examined for sets of 5000 consecutive primes through the first three million. The results reported are based upon frequency distributions (59 in total) that are calculated at intervals of 50,000 primes for $\pi(n)$ between 95,000 and 3,000,000. The quantities that are investigated include the means and standard deviations of the 59 distributions, together with coefficients that are obtained from exponential functions fitted by least-squares to the "poles" of the underlying density functions. The resulting vector of 59 estimated coefficients is then in turn related (via a least-squares regression equation) to the logarithm of p_n.

Key results of the analyses are as follows:

(1). That the *mean* of z_n increases with the logarithm of p_n is clearly confirmed.

(2). The *support* for z_n increases very slowly through the first three million primes, as the maximum z_n in the "samples" of 5000 consecutive primes that have been analyzed is never found to be larger than 178.

(3). "*Poles*" in the distribution of z_n are present at values of z_n divisible by six. These "poles" have an analytical basis, and appear to decline exponentially.

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I. Introduction

The results reported in this note refer to the distribution of $z_n = p_n - p_{n-1}$ for the first three million prime numbers (p). Key results are as follows:

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The analyses of the note are almost purely statistical.[1] The difference between consecutive prime numbers is treated as a random variable, and empirical frequency distributions are examined for sets of 5000 consecutive primes through the first three million. The results reported are based upon frequency distributions (59 in total) that are calculated at intervals of 50,000 primes for $\pi(n)$ between 95,000 and 3,000,000. The quantities that are investigated include the means and standard deviations of the 59 distributions, together with coefficients that are obtained from exponential functions fitted by least-squares to the "poles" of the underlying density functions. The resulting vector of 59 estimated coefficients is then in turn related (via a least-squares regression equation) to the logarithm of p_n.

The density function in Figure 1, for $\pi(n)$ between 995,000 and 1,000,000, is typical of the density functions for all the 59 samples that are analyzed. The "poles" (noted above) at differences divisible by six are clearly evident, as is also suggestion of their exponential decline. The basic stability of the distributions is brought home in Figure 2, which superimposes the density functions for $\pi(n)$ between 95 and 100k, 1950 and 1955k, and 2995 and 3000k. The poles are again evident, as is their suggested exponential declines. Note, however, that the distributions become flatter as $\pi(n)$ becomes larger; this is reflection, of course, of the fact (to be shown empirically below) that the mean of z_n increases with p_n.

[1] Files for the first 5.8 million prime numbers have been obtained from http://www.geocities.com/primes_r_us/small/index.html. Statistical and graphical analyses have been done in SAS and Excel.

Figure 1

Density Function for z_n
$995{,}000 < \pi(n) \leq 1{,}000{,}000$

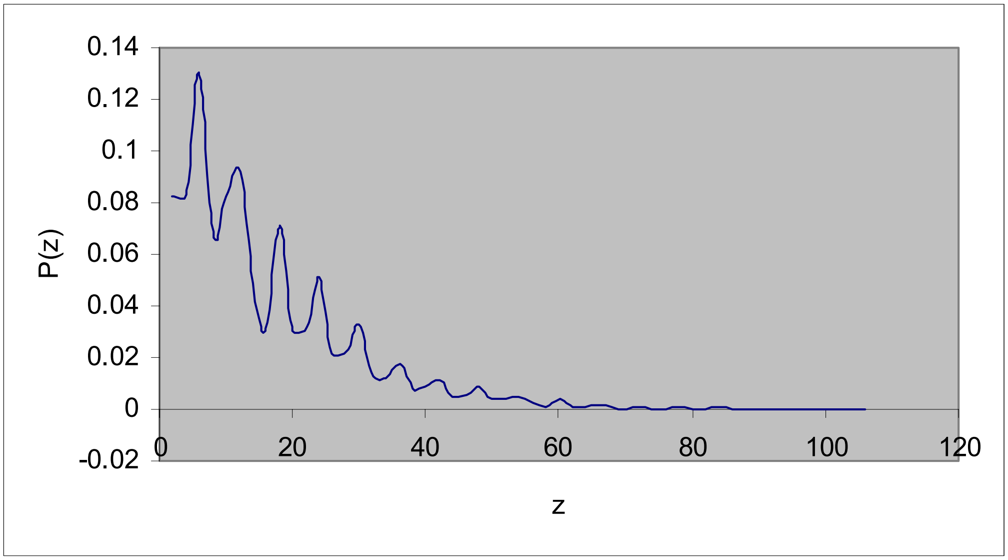
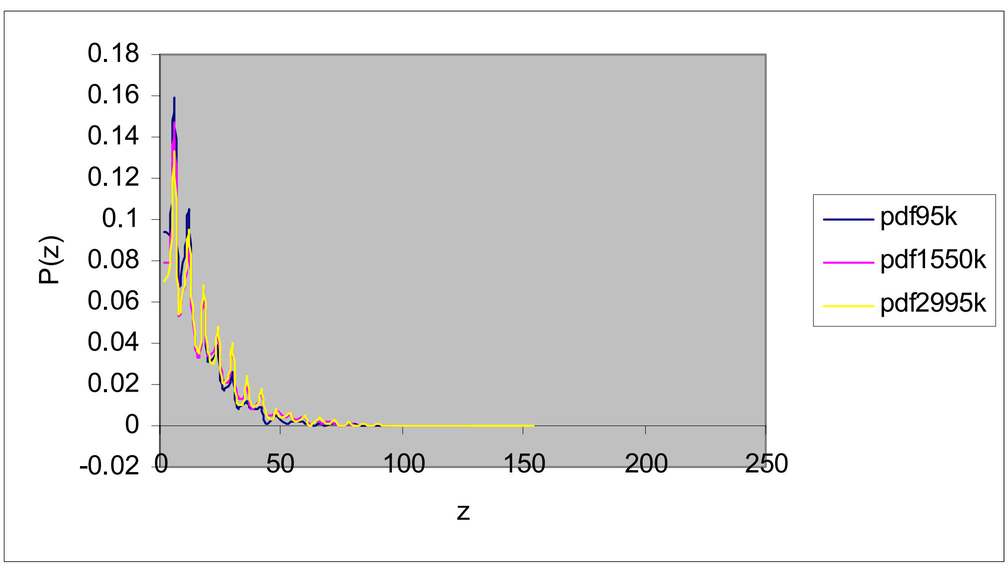


Figure 2

Density Functions for z_n
$95{,}000 < \pi(n) \leq 100{,}000$
$1{,}550{,}000 < \pi(n) \leq 1{,}555{,}000$
$2{,}995{,}000 < \pi(n) \leq 3{,}000{,}000$



II. An Important Analytical Result

The "music of the primes" has beguiled mathematicians since ancient times, indeed, so much so that proper proof of the Prime Number Theorem (in the form of the Riemann Hypothesis) continues to be viewed, as has been the case for more than 100 years, as the single most important unsolved problem in mathematics. However, while the literature related to the behavior of $\pi(n)$ is large, this is not the case for the distribution of the difference between consecutive primes. In fact, the only analytical results concerning z_n that I have been to find in the literature are (1) that the mean of z_n increases according to the logarithm of p_n and (2) that z_n tends to infinity as $O(p_n^{1/2}\ln p_n)$.[2] "Poles" on the distribution of z_n at values of z_n divisible by 6 appears not to have been noticed. This result will now be shown to have an analytical basis.

To begin with, we note that, excepting 2, all primes are necessarily odd, which in turn means that z_n must necessarily be even. However, since numbers ending in 5 are obviously divisible by 5, the only prime number that can end in 5 is 5 itself, which is to say that prime numbers (after 5) cannot end in 5. The key to the result turns on the fact that primes must necessarily be of the form $6n \pm 1$ (for n = 0, 1, 2, ...).[3] From this, it follows that values of p_n of the form 6n avoid numbers ending in 5 with greater frequency than differences of the form jn, for j = 1, ..., 5. The argument underlying this conclusion is as follows:

(1). The ways to get a z_n equal to 2, and avoid a number ending in 5, are for p_n to be equal to 6n + 1 and for p_{n-1} to be equal to 6n - 1 for 6n ending in 0, 2, or 8.

(2). The ways to get a z_n equal to 4, and avoid a number ending in 5, are for p_n to be equal to 6(n + 1) - 1 and p_{n-1} to be equal to 6n + 1 for 6n ending in 2,6, or 8.

(3). The ways to get a z_n equal to 6, and avoid a number ending in 5, are for p_n to be equal to 6(n + 1) - 1 and for p_{n-1} to be equal to 6n - 1 for 6n ending in 2, 4, or 8, or alternatively for p_n to be equal to 6(n + 1) + 1 and p_{n-1} to be equal to 6n + 1 for 6n ending in 0, 2, or 6.

(4). Finally, the ways to get a z_n equal to 8, and avoid a number ending in 5, are for p_n to be equal to 6(n + 1) + 1 and p_{n-1} to be equal to 6n - 1 for 6n ending in 0, 2, or 4.

If values of 6n ending in 0, 2, 4, 6, and 8 are assumed to be equally likely, then from (1) - (4) we see that (in relative terms) there are three "chances" each of avoiding numbers ending in 5 for

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[2] See Ivic (1985, p. 299).

[3] See Havil (2000, p. 31).

z_n divisible by 2, 4, or 8, but *six* "chances" for z_n divisible by 6. This establishes the result for the first "pole" on the density function for z_n (i.e, for z_n equal to 2, 4, 6, or 8).

For the second "pole", that is, for z_n equal to 10, 12, or 14, reasoning paralleling (1) - (4) will show that there will again be three "chances" each for a z_n equal to 10 or 14 avoiding a number ending in 5, but once again *six* "chances" that a z_n of 12 will do so. The same can be shown to hold for the third "pole" (i.e., a z_n of 18 *vis-a-vis* z_n's of 16 or 20), and so on and so forth for subsequent "poles".

III. Statistical Results

Table 1 presents means, variances, and standard deviations of z_n for the 59 "samples" of 5000 primes that have been analyzed in the study. The (natural) logarithm of the mean p_n for each sample is included as well. Although the relationships are not monotonic, the means, variances, and standard deviations of z_n are all seen to be upward-trending functions of n. However, the thing that most stands out in this table is a virtual equality of the mean z_n's of the samples with the natural logarithm of the corresponding mean p_n's.

The strength of the relationship between z_n and $\ln p_n$ is evidenced in the following least-squares regression of the mean of z_n on $\ln p_n$:[4]

$$(1) \qquad \text{mean } z_n = \underset{(0.3175)}{0.0984} + \underset{(0.0181)}{0.9939} \ln p_n \qquad R^2 = 0.9798.$$

The intercept in this equation is seen (statistically) to be close to 0, while the coefficient on $\ln p_n$ is seen to be even closer to 1.[5] The R^2 of 0.98 obviously attests to a tight fit, and the residuals from the equation, as depicted in Figure 3, appear appropriately random.[6] That the mean of z_n increases in line with the logarithm of p_n as n becomes large, in short, seems a solid conclusion.

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[4] Estimated standard errors for the regression coefficients are given in parentheses.

[5] Hypotheses (using t-tests) that the intercept and slope coefficient are *individually* equal to 0 and 1, respectively, are clearly not rejected. However, this is not the case for the composite hypothesis (using an F-test) that the parameters are *jointly* equally to these values. The calculated F-statistic under this hypothesis is 5.397, which corresponds to a *p*-value (for 2 and 57 degrees of freedom) of about 0.995.

[6] Since the "observations" being analyzed have a meaningful natural ordering with n, the Durbin-Watson statistic (which is commonly used in testing for serial dependence in time-series data) can be employed as an indicator of a "mis-specified" functional form. The sample Durbin-Watson coefficient of 1.84 provides no evidence that this is the case.

Table 1

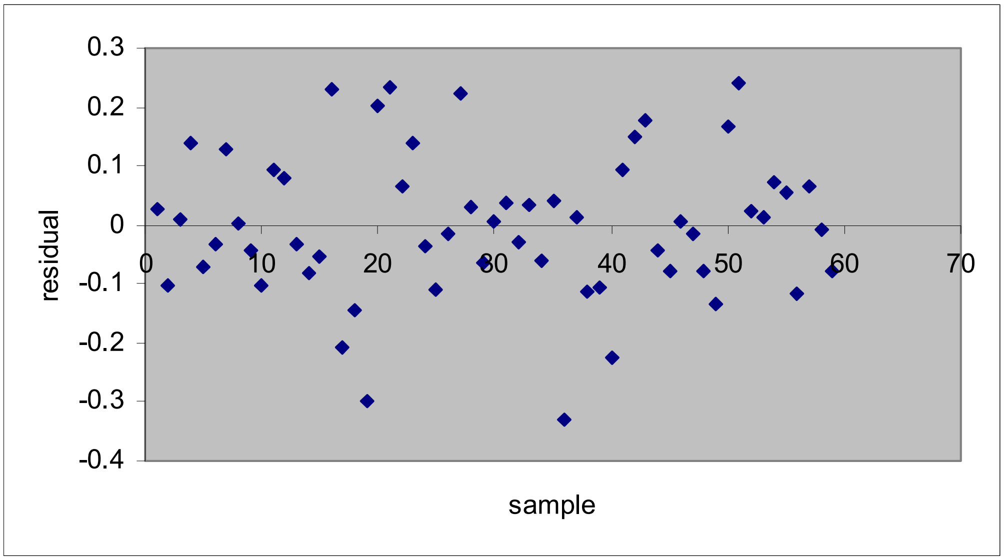
Means, Variances, and Standard Deviations of z_n
For 59 Samples of 5000 Prime Numbers

Primes (k)	LnP	Mean z_n	Var. z_n	S. D. z_n
95-100	14.05024	14.0880	136.0307	11.6632
145-150	14.49834	14.4053	133.3435	11.5474
195-200	14.81340	14.8320	150.2110	12.2561
245-250	15.05650	15.2000	159.0578	12.6118
295-300	15.25485	15.1882	149.6730	12.2341
345-350	15.42185	15.3932	164.0076	12.8065
395-400	15.56656	15.6996	166.7867	12.9146
450-455	15.70570	15.7107	170.8636	13.0715
500-505	15.81816	15.7780	177.6985	13.3304
550-555	15.92000	15.8200	173.1624	13.1591
600-605	16.01276	16.1056	174.6522	13.2156
650-655	16.09827	16.1768	183.7479	13.5554
700-705	16.17746	16.1460	177.2122	13.3121
750-755	16.25107	16.1668	176.9006	13.3004
800-805	16.31986	16.2640	184.3686	13.5782
850-855	16.38455	16.6124	185.5610	13.6221
900-905	16.44559	16.2372	178.4534	13.3586
950-955	16.50329	16.3569	182.4793	13.5085
1000-1005	16.55806	16.2565	186.2879	13.6487
1050-1055	16.61011	16.8100	195.8434	13.9944
1100-1005	16.65967	16.8888	203.5194	14.2660
1150-1155	16.70695	16.7684	204.7605	14.3095
1200-1205	16.75238	16.8888	203.5194	14.2660
1250-1255	16.79597	16.7554	190.2803	13.7942
1300-1305	16.83778	16.7235	196.9492	14.0339
1350-1355	16.87799	16.8596	201.2554	14.1865
1400-1405	16.91676	17.1356	203.5197	14.2660
1450-1455	16.95412	16.9776	217.7899	14.7577
1500-1505	16.99028	16.9220	197.6167	14.0576
1550-1555	17.02516	17.0244	201.1063	14.1812
1600-1605	17.05908	17.0918	203.7161	14.2729
1650-1655	17.09182	17.0562	204.2321	14.2910
1700-1705	17.12362	17.1529	203.1243	14.2522
1750-1755	17.15449	17.0879	201.8768	14.2083
1800-1805	17.18448	17.2202	211.5615	14.5452
1850-1855	17.21367	16.8780	202.7705	14.2397
1900-1905	17.24195	17.2479	201.7115	14.2025
1950-1955	17.26968	17.1495	216.8600	14.7262
1995-2000	17.29397	17.1793	196.2782	14.0099
2045-2050	17.32030	17.0874	201.2404	14.1859
2095-2100	17.34603	17.4334	216.3384	14.7084
2145-2150	17.37122	17.5112	226.4886	15.0495
2195-2200	17.39575	17.5652	224.1084	14.9703
2245-2250	17.41973	17.3696	214.5455	14.6474
2295-2300	17.44316	17.3572	213.1882	14.6010

Primes (k)	LnP	Mean z_n	Var. z_n	S. D. z_n
2345-2350	17.46607	17.4636	206.9737	14.3866
2395-2400	17.48845	17.4664	225.2959	15.0099
2445-2450	17.51044	17.4228	216.2821	14.7065
2495-2500	17.53196	17.3880	208.2722	14.4316
2545-2550	17.55313	17.7120	228.4204	15.1136
2595-2600	17.57381	17.8068	233.8729	15.2929
2645-2650	17.59415	17.6072	221.2797	14.8755
2695-2700	17.61406	17.6176	227.4317	15.0808
2745-2750	17.63354	17.6964	236.3292	15.3730
2795-2800	17.65275	17.6968	232.7254	15.2553
2845-2850	17.67159	17.5440	213.5602	14.6137
2895-2900	17.69017	17.7456	217.2307	14.7387
2945-2950	17.70840	17.6896	221.8107	14.8933
2995-3000	17.72624	17.6396	218.0727	14.7673

Figure 3

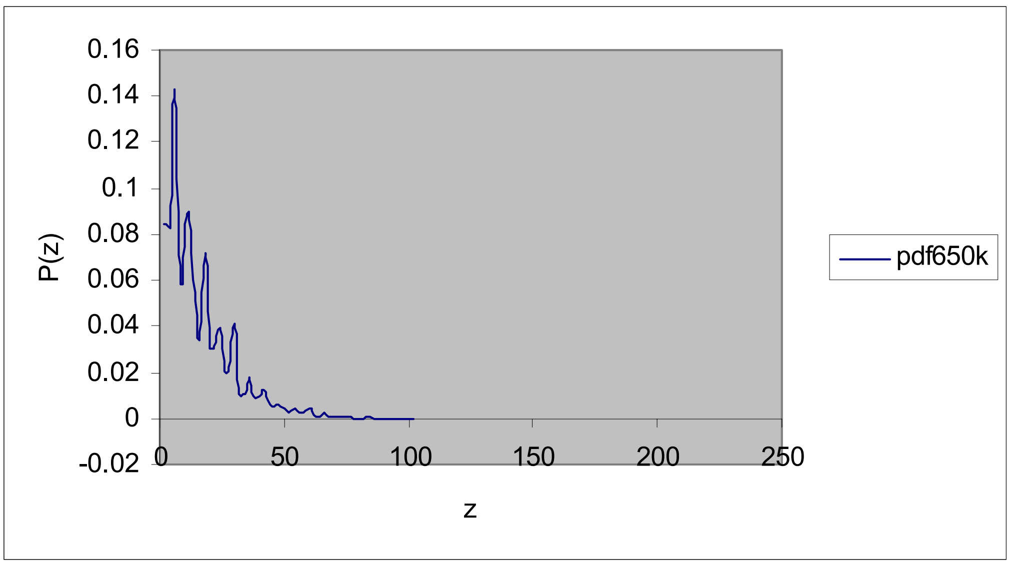
Scatter Diagram of Residuals from Equation (1)



Let us now turn our attention to the apparent exponential decline in the "poles" on the density functions of z_n at values of z_n that are divisible by 6. However, lest it be thought that such a decline (as with the existence of the poles themselves) might be analytical, the graph in Figure 4 for $\pi(n)$ between 650,000 and 655,000 shows that this is not the case, for the pole for z_n equal to 30 on this density function is higher even than the pole corresponding to z_n equal to 24. Such "anomalies" are not uncommon, especially in the poles after the fourth, (that is for z_n's of 24). Accordingly, possible exponential decline in the poles can only be seen as a statistical phenomenon.

Figure 4

Density Function for z_n
$650{,}000 < \pi(n) \leq 655{,}000$



In investigating statistically whether the decline may in fact be exponential, functions of the form,

$$\ln P(z) = \alpha + \beta z, \tag{2}$$

have been fitted by least-squares to the first 8 poles (i.e., for z = 6, 12, 18, 24, 30, 36, 42, 48) for each of the 59 probability density functions that have been estimated.[7] The resulting estimates of β, associated t-statistics, and R^2's for the 59 equations are tabulated in Table 2.

[7] Only the first 8 poles are analyzed because probability masses on z_n greater than 48 are less than 0.01. While it seems reasonable to suppose that the poles at values z_n divisible by 6 will eventually stabilize, $\pi(n)$ of many millions (or even billions) may be required.

Table 2

Estimated Exponential Functions

Primes (k)	β	t-ratio	R^2
95-100	-0.08505	-32.28	0.9943
150-155	-0.08077	-19.24	0.9840
200-205	-0.07986	-19.03	0.9837
250-255	-0.07559	-30.70	0.9937
300-305	-0.07595	-15.89	0.9768
350-355	-0.07179	-20.24	0.9856
395-400	-0.07251	-12.51	0.9631
450-455	-0.07003	-13.76	0.9693
500-505	-0.06967	-18.20	0.9822
550-555	-0.07179	-15.36	0.9752
600-605	-0.07175	-13.81	0.9695
650-655	-0.07235	-13.07	0.9661
700-705	-0.07075	-16.42	0.9782
750-755	-0.06820	-21.77	0.9854
800-805	-0.07006	-20.52	0.9859
850-855	-0.06758	-15.16	0.9746
900-905	-0.06712	-20.00	0.9852
950-955	-0.07121	-19.64	0.9847
1000-1005	-0.06561	-17.73	0.9813
1050-1055	-0.06449	-14.69	0.9729
1100-1005	-0.06533	-15.24	0.9748
1150-1155	-0.06533	-15.24	0.9706
1200-1205	-0.06784	-12.05	0.9603
1250-1255	-0.06741	-28.06	0.9924
1300-1305	-0.06457	-14.20	0.9711
1350-1355	-0.06330	-45.18	0.9971
1400-1405	-0.06520	-16.74	0.9790
1450-1455	-0.06818	-16.51	0.9785
1500-1505	-0.06439	-16.97	0.9762
1550-1555	-0.06609	-16.88	0.9794
1600-1605	-0.06432	-20.49	0.9859
1650-1655	-0.06287	-16.96	0.9796
1700-1705	-0.06954	-15.40	0.9753
1750-1755	-0.06609	-16.41	0.9782
1800-1805	-0.06706	-29.66	0.9932
1850-1855	-0.06722	-20.70	0.9862
1900-1905	-0.06440	-16.95	0.9795
1950-1955	-0.06549	-16.17	0.9776
1995-2000	-0.06259	-14.54	0.9724
2045-2050	-0.06680	-15.96	0.9770
2095-2100	-0.06059	-24.21	0.9899
2145-2150	-0.06296	-25.81	0.9911
2195-2200	-0.06494	-12.83	0.9649
2245-2250	-0.06194	-16.92	0.9795
2295-2300	-0.06356	-20.11	0.9829
2345-2350	-0.06145	-13.68	0.9689
2395-2400	-0.06588	-13.69	0.9690

Primes (k)	β	t-ratio	R^2
2445-2450	-0.06782	-14.47	0.9721
2495-2500	-0.06174	-17.98	0.9818
2545-2550	-0.06300	-16.11	0.9774
2595-2600	-0.06210	-20.63	0.9861
2645-2650	-0.06209	-21.78	0.9875
2695-2700	-0.06086	-19.72	0.9848
2745-2750	-0.06559	-20.14	0.9854
2795-2800	-0.06076	-20.79	0.9863
2845-2850	-0.06036	-23.64	0.9894
2895-2900	-0.05967	-17.46	0.9807
2945-2950	-0.06541	-16.32	0.9780
2995-3000	-0.06154	-16.36	0.9781

Despite the fewness of degrees of freedom, the equations seem to fit the "observations" well. R^2's, typically of the order of 0.98, are never less than 0.96, while t-ratios for the estimated exponential parameter (β) are generally of the order of -16 to -18. The estimated exponential parameters themselves decline (in absolute value), more or less continually with π(n), from a value of -0.0851 for π(n) between 95,000 and 100,000 to -0.0615 for π(n) between 2,995,000 and 3,000,000.

In view of the known close relationships between π(n) and the mean of z_n and the logarithm of p_n, it is natural to enquire into whether there might also be a relationship between the estimated exponential parameters in Table 2 and ln p_n. A plot of the two quantities is given in Figure 5. Two results are visible in the plot. The first is a "heteroscedasticity" in the estimated exponential parameters, in that the "variance" of the estimated parameters (viewed as a random variable) quite clearly increases with the value of ln p_n, while the second result is a subtle (yet unmistakable) hint that the relationship between the exponential parameters and ln p_n is non-linear. In view of the latter, a plausible next step (but still keeping with logarithms) is to seek out the relationship between the estimated exponential parameters and the *logarithm* of the logarithm of p_n. The plot of these quantities is given in Figure 6. Although "heteroscedasticity" continues to be in evidence, a much more linear relationship is now apparent.

We turn now to a least-squares regression analysis of the relationships in Figures 5 and 6, in which the estimated exponential parameters from column 2 of Table 2 are regressed on ln p_n and ln(ln p_n). The resulting equations are as follows:[8]

$$(3) \qquad y = \underset{(-29.93)}{-0.1588} + \underset{(17.34)}{0.00548} \ln p_n \qquad R^2 = 0.8406$$

$$(4) \qquad y = \underset{(-22.51)}{-\,0.3180} + \underset{(17.78)}{0.0891} \ln(\ln p_n) \qquad R^2 = 0.8472\ ,$$

[8] This time, the quantities in parentheses are t-ratios.

Figure 5

Relationship Between Estimated Exponential Parameters in Table 2
And The Logarithm of Corresponding Mean p_n

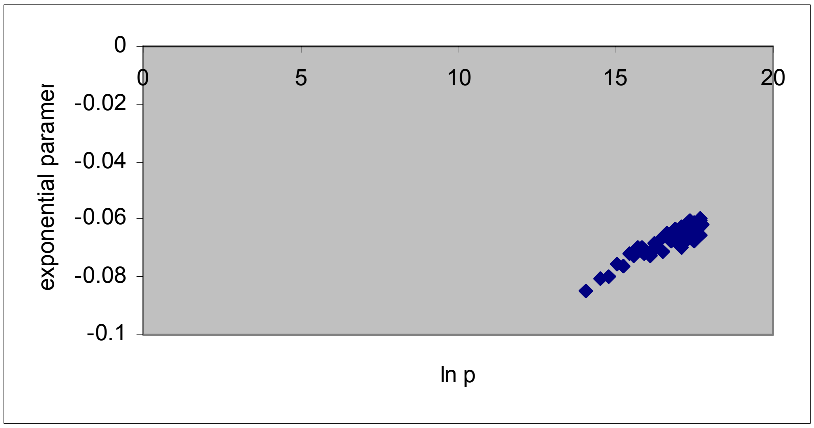
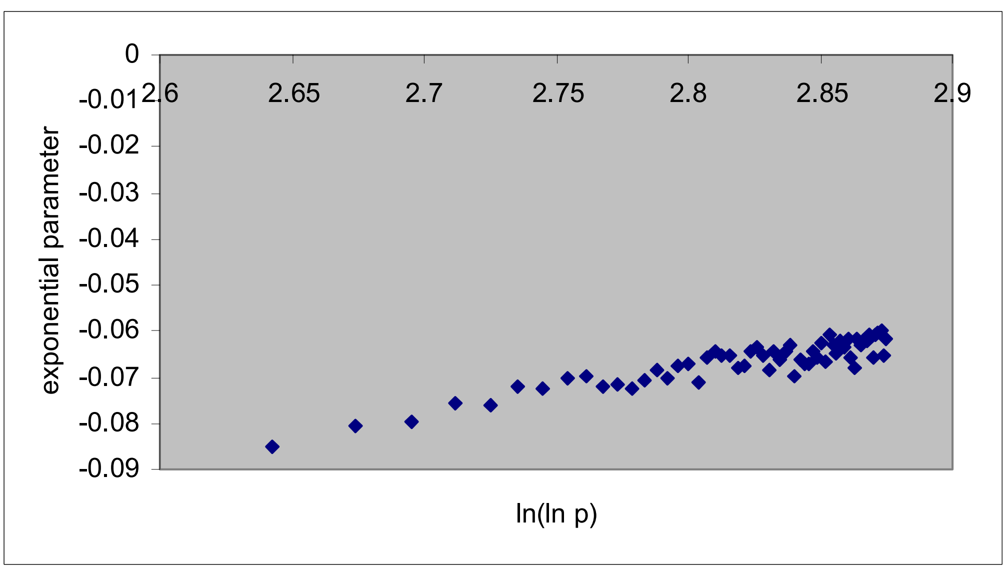


Figure 6

Relationship Between Estimated Exponential Parameters in Table 2
And The Logarithm of The Logarithm of Corresponding Mean p_n



where y denotes the vector of values for the estimated exponential parameters from column 2 of Table 2. As is expected, the coefficients on the logarithmic functions of p_n are seen to be positive, with corresponding *p*-values that are all less than 0.0001.

In view of the "heteroscedasticity" displayed in Figures 5 and 6, statistical efficiency can be improved if the equations are estimated by "weighted" (i.e., generalized) least squares, rather than ordinary least squares. Weighted least-squares equations have been accordingly been estimated using as weights the square root of the products of $1/\ln p_n$ and $1/\ln(\ln p_n)$ with the absolute value of the t-ratios of the estimated exponential parameters as weights.[9] The estimated equations are as follows:

$$\text{(5)} \qquad y = \underset{(-31.83)}{-0.1607} + \underset{(18.57)}{0.00560} \ln p_n \qquad R^2 = 0.8581$$

$$\text{(6)} \qquad y = \underset{(-23.76)}{-0.3217} + \underset{(18.81)}{0.0904} \ln(\ln p_n) \qquad R^2 = 0.8613\ .$$

Although the estimated coefficients in these equations are virtually the same as the ones in equations (3) and (4), there is a noticeable improvement in fit and estimation efficiency, particularly in the model with $\ln(\ln p_n)$ as the "independent" variable.

Alll of the results to this point has been with respect to "samples" drawn from the first 3 million prime numbers. In closing this section, I would like to finish with the density functions in Figure 7 for $\pi(n)$ between 5000 and 10,000 and for $\pi(n)$ between 5,795,000 and 5,800,000. The graphs, I think, pretty much speak for themselves.

IV. Conclusions

Upon seeing the density functions depicted in Figures 1 and 2, the reaction of an econometrician colleague was, "This is the crazy!" This was my reaction at first exposure as well, for it was not until after I saw the poles in $f(z_n)$ at values of z_n divisible by 6 that I came to understand why such values are privileged -- namely, because they avoid, with greater probability, odd integers ending in 5. This combined statistical/analytical result seems quite clearly to be the most important finding of the investigation.

A second consequential finding, it seems to me, is the rather strong statistical evidence [Table 2, equation (6)] that the poles decline exponentially, but with a parameter that increases (more or less) with the log-log of p_n. If an exponential decline in poles should turn out to be an asymptotic

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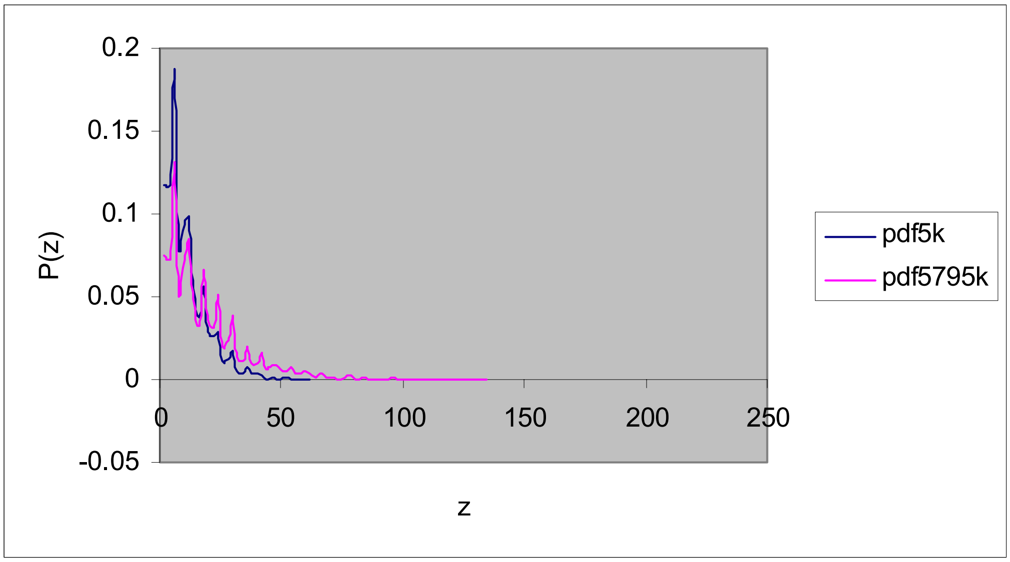
[9] The use of $1/\ln p_n$ and $1/\ln(\ln p_n)$ as weights follows straightforwardly from the "scatter" in the estimated exponential parameters pictured in Figures 5 and 6. On the other hand, since the exponential parameters are themselves measured with error, the square root of the absolute value of their associated t-ratios seems an appropriate additional weight as well.

Figure 7

Density Functions for z_n
$5000 < \pi(n) \leq 10000$
$5,795,000 < \pi(n) \leq 5,800,000$



result, then one ought to be able to conclude that *the "true" distribution of z_n, whatever it might be, is bounded by a well-defined exponential distribution*.[10] A third finding -- although it is more a confirmation -- is of the critically close relationships between prime numbers and the logarithmic function. While such relationships concerning p_n and $\pi(n)$ have been at the basis of analytical number theory since Gauss, the present investigation confirms (on a statistical basis) the relationship between the mean differences between consecutive primes and ln p_n, but also proffers, perhaps for the first time, the suggestion that there may be an equally close relationship between the distribution of these differences and ln p_n.

A final result of interest, and one that has not previous been noted, refers to the proportion of the total probability of z_n that is accounted for by z_n's that are divisible by 6. From the discussion in Section II, with prime numbers having to be of the form $6n \pm 1$, there 15 possible ways of getting a difference between odd integers that avoid an integer ending in 5, and of these 6 are associated with a difference of 6. In view of this, and under the assumption that all 15 possibilities are equally likely, then the proportion of the probability of the difference between consecutive primes being divisible by 6 should be 0.4. For $\pi(n)$ between 2,995,000 and 3,000,000, the empirical probability (for the

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[10] However, in view of the "heteroscedasticity" evidenced in Figures 5 and 6, it is best to be cautious at this point regarding any conclusion that this is in fact the case. "Well-defined" in this context is meant in the sense that the exponential parameter can [from equation (6)] be estimated as $-0.3217 + 0.0904 \ln(\ln p_n)$.

first 8 poles) is 0.448, while for $\pi(n)$ between 5795,000 and 5,800,000, the corresponding value is 0.418, suggesting the asymptotic value may indeed be 0.4.

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